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Quantum Graphics

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A graph-theoretic particle model is presented. The vertices of the graph represent elementary particles and the edges represent strong interactions. Each physical particle is described by an essentially unique labeled graph. Weak and electromagnetic interactions become manifestations of the strong interactions. Observed particles are classified according to this model and new particles are predicted. Decay processes are illustrated in the model. A representation of particles in a 48-dimensional hyperspace is given and it is indicated that this may be the proper arena in which a dynamical theory can be developed.

1. INTRODUCTION

A particle model is presented in which there is one elementary particle and one basic interaction. We call the elementary particles vertices and the basic interaction is called the strong interaction. In this model, every particle is composed of vertices in a certain configuration and all interactions are manifestations of the strong interaction (we do not consider gravity in this work, although there is a good possibility that it can be included later). Each vertex is a spin-$\frac{1}{2}$ fermion and maintains an electric and color charge. The properties of a particle are determined by its graphical configuration and by the spins and charges of its vertices.

In this work we mainly consider the qualitative features of the model. We first develop a classification scheme for particles in terms of their constituent vertices. This scheme includes all observed physical particles (Aguilar-Benitez et al., 1984). The allowed graphical configurations of the particles are determined by a few physically motivated rules. The constituents of hadrons are similar to that of the usual quark model (Close, 1979) up through the level of charmed hadrons. Above this level, however, the two models diverge in their predictions. Next we give examples of various decay processes in terms of changing graphical configurations.

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Finally, we set the stage on which a dynamical theory may possibly be developed. This is a 48-dimensional linear space which we call hyperspace.

The present model has certain advantages over other particle models. First, there is no need to postulate different "generation" levels (three in the current quark model). These levels occur automatically and naturally in the present model. Second, one can "see" how the various particles (including leptons and photons) develop in decay and scattering processes. Third, particles are described concretely instead of as unphysical superpositions of various particles states.

We hope to eventually develop a dynamical theory for this model which will give quantitative predictions. Such a theory might be called quantum graphicdynamics, or QGD. If this program is successful, QGD would give a discrete particle model for which QED and QCD would be continuum approximations.

2. GRAPHICAL CONFIGURATIONS

We assume that there is an elementary particle, which we call a *vertex*, and that every physical particle is composed of vertices in some configuration. We also assume that there is an interaction called the *strong interaction*, which may act between pairs of vertices. If the strong interaction acts between distinct vertices u and v, we write $u \perp v$ and say that u, v are *adjacent*. If $u \perp v$, we call the pair $uv = \{u, v\}$ an *edge*. In mathematical terminology, a pair $G = (V, E)$, where V is a finite set of vertices and E is a set of edges on V, is called a *graph*. A graph is *even* (*odd*) if it has an even (odd) number of vertices.

In this paper, when we refer to a "particle" we shall mean a hadron, lepton, or gauge boson (photon or weak gauge boson). Our goal is to represent each of these particles by a unique labeled graph which describes the particle's physical properties. Before embarking on this program, we shall need some graph-theoretic definitions.

A *path* in a graph $G = (V, E)$ is a set of (not necessarily distinct) vertices $v_1, \ldots, v_n \in V$ such that $v_i \perp v_{i+1}$, $i = 1, \ldots, n-1$. Moreover, we call v_1 and v_n the *initial* and *final* vertices of the path, respectively. If the initial and final vertices are equal, the path is called a *cycle*. We say that G is *connected* if any two vertices of G are joined by a path. A path is *even* (*odd*) if it contains an even (odd) number of distinct vertices. A subgraph H of G is *complete* if any two distinct vertices u, v in H satisfy $u \perp v$. A *clique* in G is a complete subgraph of G. A clique K is an *n-clique* if its cardinality $|K| = n$.

We assume that every particle is described by a connected graph. Since there is a great variety of connected graphs, we must impose certain physical rules to distinguish those that correspond to physical particles. First, there appear to be two types of vertices, valence vertices and interaction vertices. The valence set (i.e., the set of valence vertices) consists of the basic building blocks of a particle and these vertices all interact strongly with each other, while the interaction set describes weak interactions between valence vertices (they also describe electromagnetic interactions, but these will be considered later). It follows that the valence set forms a clique in the connected graph representing a particle.

We next assume that every vertex is a spin-$\frac{1}{2}$ fermion. Thus each vertex has spin $\frac{1}{2}$ (denoted $\uparrow$) or spin-$\frac{1}{2}$ (denoted $\downarrow$) and the total spin of a particle is the sum of the spins of its constituent vertices. It follows that fermions (leptons, baryons) have an odd number of vertices and bosons (mesons, gauge bosons) have an even number of vertices. Since the spin of a particle should be determined by its valence clique, we conclude that the valence clique of a fermion is odd and the valence clique of a boson is even. It follows that the interaction set for any particle is even. These properties are summarized in the following configuration rule.

Rule (C). Leptons and baryons are represented by odd connected graphs containing an odd valence clique, while mesons and gauge bosons are represented by even connected graphs containing an even valence clique.

We now make Rule (C) more precise. Since leptons appear to be "pointlike," structureless particles, they should have the simplest valence clique. We thus assume that the valence clique of a lepton has just one element. We next assume that the valence clique of a baryon is an odd n-clique with $n \geq 3$. If $n = 3$, we say that the baryon is in its *ground state.* The valence clique of a meson is an even n-clique with $n \geq 2$, and if $n = 2$, the meson is in its *ground state.* Since photons mediate the electromagnetic interaction, we assume that all their vertices are interaction vertices, so their valence clique is empty. In the next section we shall assume that the charge of a particle is determined by its valence clique. Thus, although weak bosons mediate the weak interaction, they can have electric charge, so their valence clique cannot be empty. We assume that the valence clique of a weak gauge boson is a 2-clique.

Since the total spin of a particle is determined by the spins of its valence vertices, the spins of its interaction vertices must cancel. The simplest way for this to be achieved "locally" is to assume the following spin rule.

Rule (S1). Adjacent interaction vertices have opposite spin.

Of course, Rule (S1) does not guarantee, at this point, that the spins of interaction vertices cancel, but this will follow from (S1) and later rules. We call a path $v_1, \ldots, v_n$ where the initial and final vertices are distinct

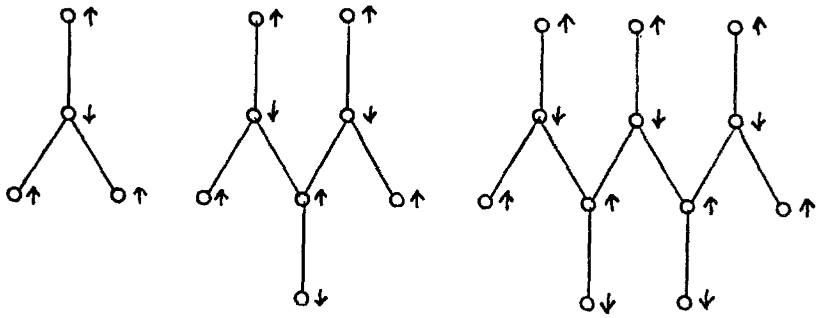

Fig. 1. Photons.

valence vertices and the others are interaction vertices an *interaction path* (it is assumed that there are interaction vertices in such a path). If n is even (odd), we call the edge v_1v_n a + (−) *interaction.* Experiments indicate that weak interactions between valence vertices are spin dependent. The simplest way to incorporate this is the following rule, which is quite similar to (S1):

Rule (S2). If $v_1, \ldots, v_n$ is an interaction path, then v_1, v_2 and v_{n-1}, v_n have opposite spin.

It follows from (S2) that if uv is a − (+) interaction, then the spins of u and v agree (are opposite). Of course, uv cannot be both a + and − interaction.

We know that the spin of a photon is 1. The simplest configurations satisfying the above criteria in which the spin *must* be 1 are shown in Figure 1. We take these to represent photons of increasing wavelength.

3. CHARGE

We make the following assumptions concerning electric charge. The electric charge of a particle is determined by its valence clique and all interaction vertices have electric charge 0. Of course, the total electric charge must be an integer. The valence vertex for a massive lepton has electric charge ±1. The valence vertex of a neutrino has electric charge 0. The valence vertices of hadrons have electric charge ±1/3, ±2/3. It follows that all vertices of a photon have electric charge 0. The weak gauge bosons are illustrated in Figure 2.

Fig. 2. Weak gauge bosons.

Let V be the vertex set for a graph which represents a particle. A function $f: V \to \mathbb{R}$ is called a *color function* if there exists a number $\hat{f} \in \mathbb{R}$ such that

$$\frac{1}{|K|} \sum_{v \in K} f(v) = \hat{f}$$

where K is the valence clique (if $K \neq \emptyset$) or any edge joining two interaction vertices. A color function describes a distribution of color charges on the vertices and $\hat{f}$ corresponds to "white" or "colorless." Thus, the valence clique has average color white and is colorless. Similarly, any interaction edge is colorless. We say that two vertices u, v have the same *color state* if $f(u) = f(v)$ for any color function f. There always exist color functions, since any constant function is one. As we shall see later, most particles possess nonconstant color functions. Notice that if uv is a $-$ interaction, then u and v have the same color state, while if uv is a $+$ interaction, then $f(u) = 2\hat{f} - f(v)$ for any color function f.

We say that a valence vertex v is *flowered* if v is contained in an odd cycle in which all the other vertices are interaction vertices. If v is flowered, it is easy to see that $f(v) = \hat{f}$ for any color function f, so v is colorless. We think of flowered vertices as those that are involved in a weak self-interaction.

We say that two valence vertices u, v are in the same *interaction state* when u is in a $+$ interaction ($-$ interaction, flowered) if and only if v is in a $+$ interaction ($-$ interaction, flowered). Even if u and v are in the same interaction state, their interaction configurations may be different. In Figure 3a two vertices are flowered, but the flowers are different. Figures 3b and 3c show similar possibilities for $+$ interactions and $-$ interactions, respectively. In these figures we denote valence vertices by open circles and interaction vertices by dark circles. We say that two valence vertices u, v have the same *configuration state* if u and v have the same interaction state and the interactions are identical.

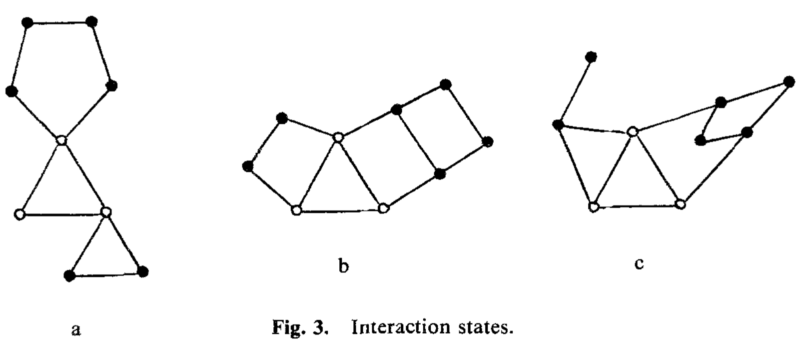

Fig. 3. Interaction states.

We are now ready to present the remaining rules. The following rule is the Pauli exclusion principle and the subsequent rule is clear.

Rule (P). Two valence vertices in a single particle cannot have the same spin, electric charge, color state, and configuration state.

Rule (S3). If the spins of all vertices composing a particle are reversed, the particle is unaltered.

The next rule states that all (weak) interactions are given by those already considered.

Rule (1). An interaction vertex of a lepton or hadron is contained in a $\pm$ interaction or a flower.

The last two rules are similar to symmetry and antisymmetry principles for wave functions.

Rule (2). If two valence vertices are the only vertices in a certain interaction state, then interchanging their spins does not alter the particle.

Rule (3). If two valence vertices have the same electric charge and interaction state, then their spins are equal.

4. DIAGRAMS

Using the rules developed in previous sections, we now have a manageable number of allowed particle graphs. We can now represent each particle by an essentially unique allowed labeled graph or diagram, and conversely each allowed diagram should correspond to a unique particle. The diagram of a particle is obtained by drawing a corresponding graph in which each vertex is labeled by its spin and electric charge. The space of color functions is determined by the graph, so no color charge labels are necessary. To avoid complicated figures, we frequently only draw the valence clique, indicate flowered vertices by dark circles, unflowered vertices by open circles, + interactions by plus signs, and $-$ interactions by minus signs. We also frequently represent cliques by straight line paths. Figure 4 illustrates two equivalent diagrams where we have omitted the vertex labels for simplicity.

The gauge bosons are already diagrammed in Figures 1 and 2. To save space, we only diagram particles, the corresponding antiparticles being obtained by taking the negative of all electric charges. We also frequently omit the 0 for zero electric charges. Figure 5 illustrates the diagrams for the leptons. One reason why τ^- has three petals and not two is given in Section 6.

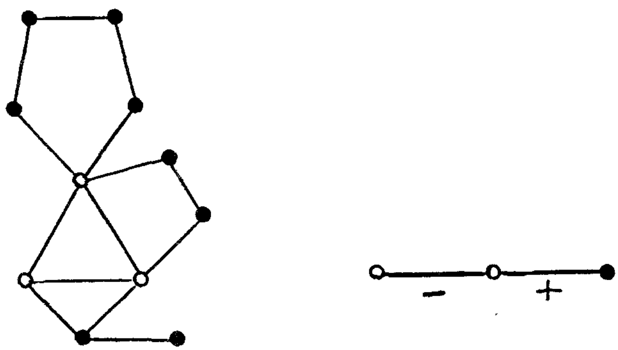

Fig. 4. Equivalent diagrams.

Figure 6 illustrates the diagrams for the ground-state mesons. Notice that flowered, electric charge ±1/3 vertices correspond to strange quarks, flowered, electric charge ±2/3 vertices correspond to charmed quarks, and ± interactions correspond in some sense to bottom quarks. Although it cannot be seen in the diagram, it is assumed that flowered, electric charge

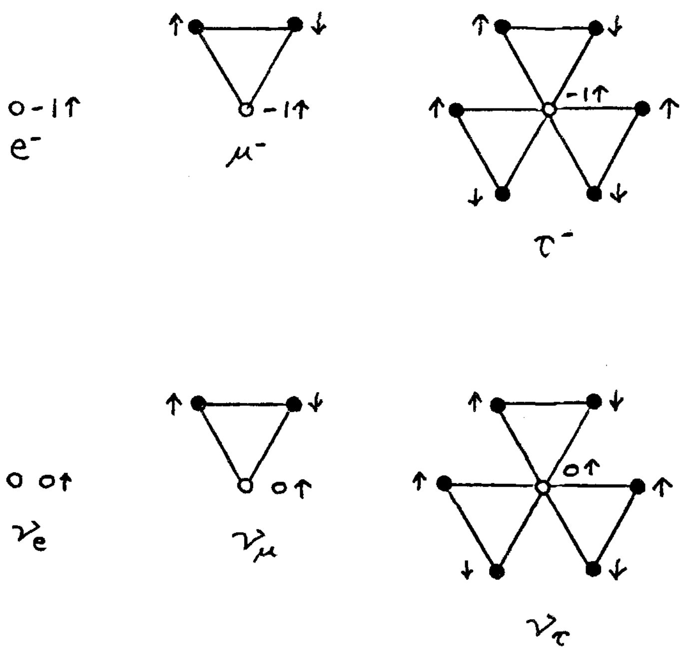

Fig. 5. Leptons.

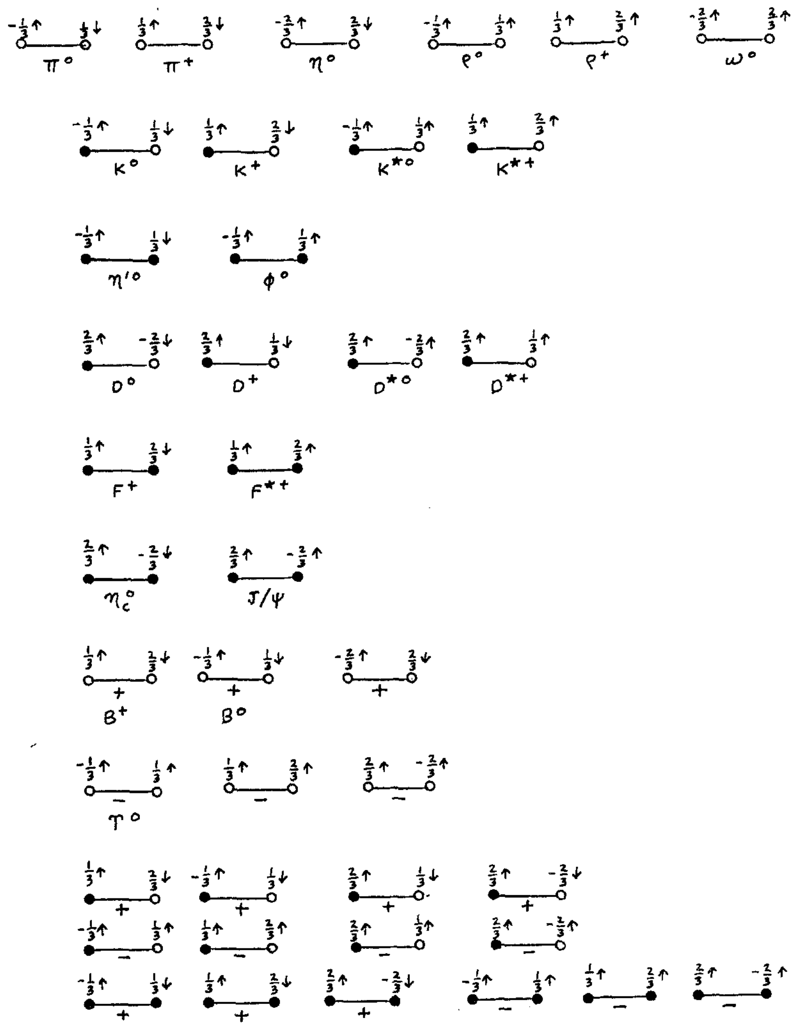

Fig. 6. Mesons.

±1/3 vertices have one petal, while flowered, electric charge ±2/3 vertices have three petals. The classifications of this model seem to correspond to the usual quark model up to the η_c, J/ψ level, but then the two models diverge. The present model predicts that there can be no spin-1 ground-state B meson and no spin-0 ground-state Υ particle, while the quark model predicts such particles.

We propose that the excited-state diagrams are obtained from the ground-state diagrams by the addition of (1/3, −1/3) and (2/3, −2/3) pairs

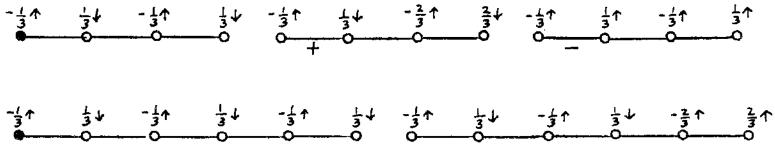

Fig. 7. Excited states.

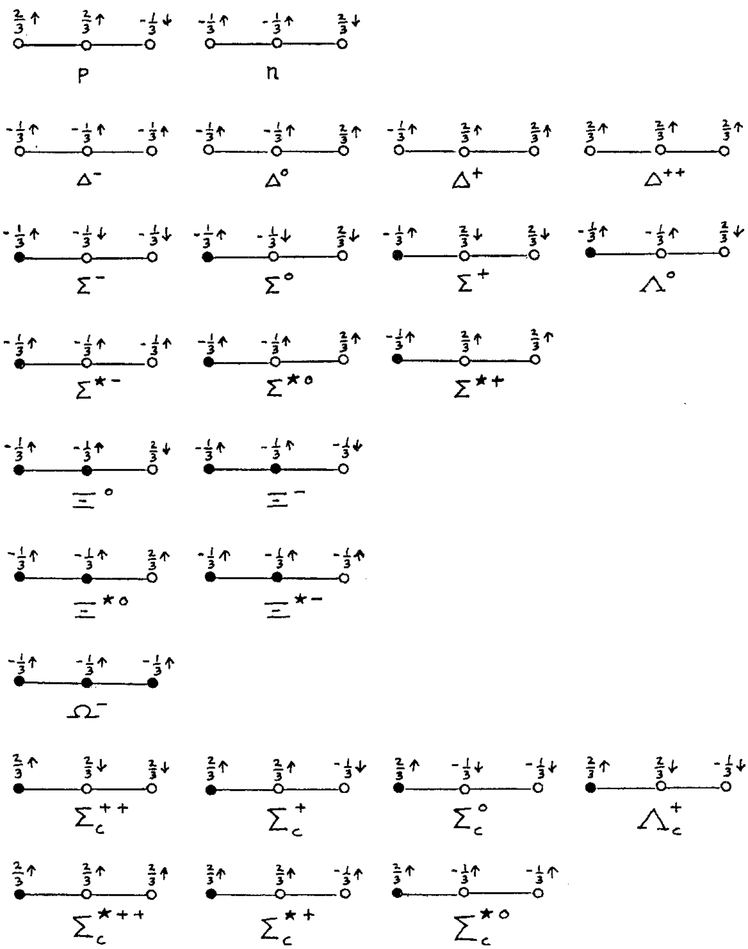

Fig. 8. Baryons.

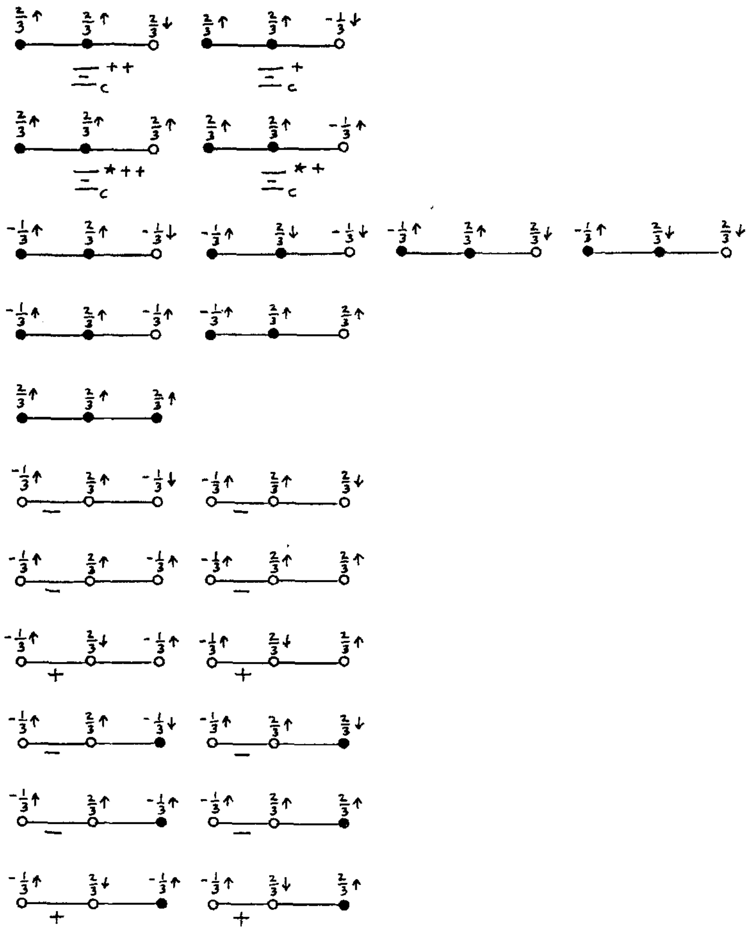

Fig. 8. Continued.

of valence vertices. These additional pairs of vertices are not contained in any (weak) interactions. Figure 7 illustrates a few excited states.

We now turn to the ground-state baryons, which are illustrated in Figure 8. Due to the large number of ground-state baryons and the fact that most of them have not yet been observed, we do not illustrate them all (we have omitted those in which vertices are in more than one interaction). However, all the allowed diagrams up to this level are included. Notice, for example, that Ξ^0 does not violate the Pauli exclusion principle, since we can assume that the two flowers are different. A similar observation holds for ± interactions. Again the classifications of the present model and the usual quark model agree up through the charmed level and then diverge.

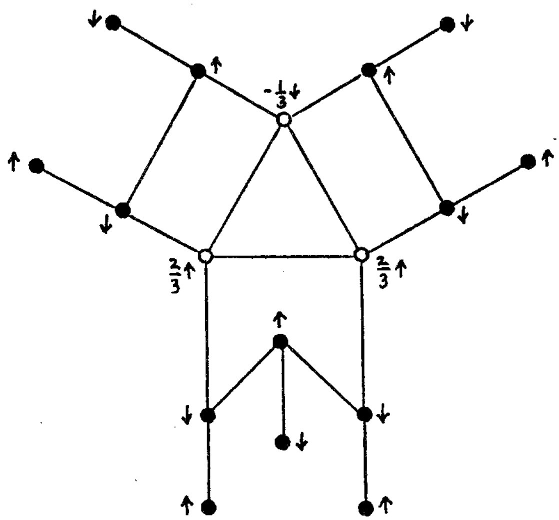

Fig. 9. Proton.

As before we propose that excited-state diagrams are obtained from the ground states by the addition of $(1/3, -1/3)$ and $(2/3, -2/3)$ pairs of valence vertices.

Until now we have not considered electromagnetic interactions. We assume that these are independent of weak interactions and are given by photonlike interaction paths which do not carry color. We do not need to include them in the diagram of a particle, since their configurations are already determined by the spins (and their strength by the electric charge) of the interacting valence particles. Figure 9 illustrates the electromagnetic interactions for the proton.

5. DECAY PROCESSES

In this section we illustrate some decay processes in terms of diagrams. The "decay" of a diagram proceeds according to the following rules.

Rule (D1). Flowers and ± interactions can be formed to produce forbidden diagrams.

Rule (D2). Electric charge and spin can redistribute as long as the total electric charge remains constant.

Rule (D3). Edges can be created and destroyed as long as the baryon number remains constant.

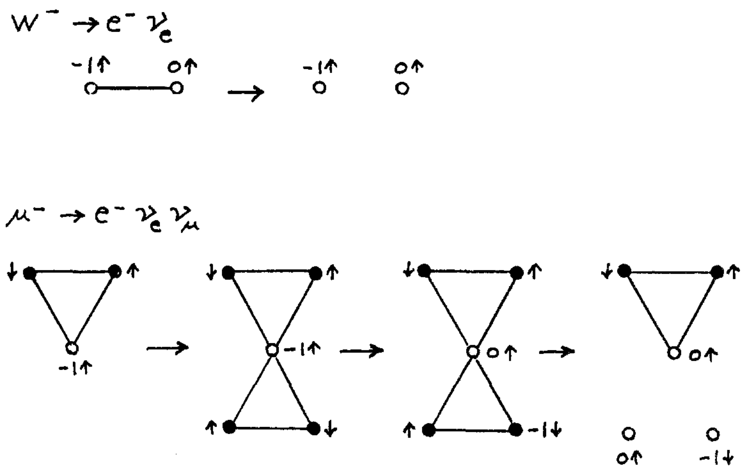

Fig. 10. Decay processes.

Rule (D4). Final diagrams must be allowed diagrams.

Of course, various decay modes are usually possible. It is hoped that once the dynamics of the theory is developed, the possible decay modes and their probabilities can be predicted. Figures 10–12 illustrate a few decay processes.

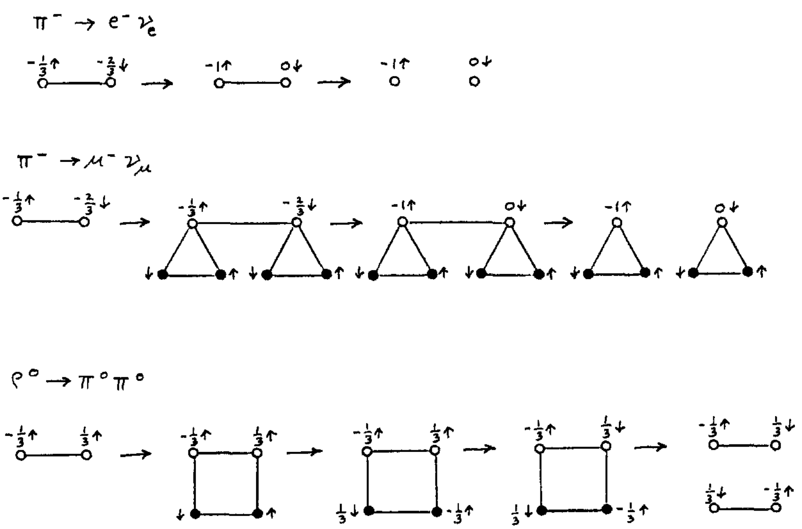

Fig. 11. Decay processes.

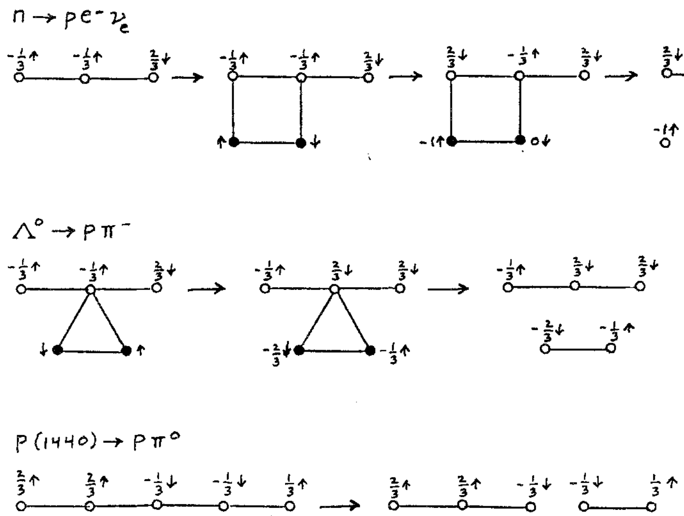

Fig. 12. Decay processes.

6. HYPERSPACE REPRESENTATION

In this section we represent particles by vectors in a 48-dimensional linear space which we call hyperspace. This may be the appropriate space on which a dynamical theory can be developed.

We first define a color space representation. Let $G=(V,E)$ be a graph representing a particle and let V_G be the linear space of color functions on G. A set of valence vertices $\{v_1,\ldots,v_n\}$ is called *V_G-dense* if $f, g\in V_G$ and $f(v_i)=g(v_i)$, $i=1,\ldots,n$, imply $f=g$. A V_G-dense set is *minimal* if it does not contain a strictly smaller V_G-dense set. Let $\{v_1,\ldots,v_n\}$ be a minimal V_G-dense set. It can be shown that $\dim V_G=n$ (Gudder and Rüttimann, 1986). Moreover, there exist unique elements $f_1,\ldots,f_n\in V_G$ such that $f_i(v_j)=\delta_{ij}$, $i,j=1,\ldots,n$, and these elements form a basis for V_G (Gudder and Rüttimann, 1986). It follows that any $f\in V_G$ has the unique representation

$$f=\sum_{i=1}^{n} f(v_i)f_i$$

For $f,g\in V_G$, define the inner product

$$\langle f,g\rangle=\sum_{i=1}^{n} f(v_i)g(v_i)$$

Then V_G becomes a real inner product space with orthonormal basis

$\{f_1, \ldots, f_n\}$. For $v \in V$, define $v^* \in V_G$ by

$$v^* = \sum_{i=1}^{n} f_i(v) f_i$$

It follows that $f(v) = \langle f, v^* \rangle$ for any $f \in V_G$, and for $u, v \in V$ we have

$$\langle u^*, v^* \rangle = \sum_{i=1}^{n} f_i(u) f_i(v)$$

We call the above representation the *color space representation* of G. We obtain an equivalent representation on $\mathbb{R}^n$ by defining the unitary transformation T: $V_G \to \mathbb{R}^n$ given by

$$Tf = (f(v_1), \ldots, f(v_n))$$

We then have

$$Tv^* = (f_1(v), \ldots, f_n(v))$$

In order to represent a vertex $v \in V$ by a unit vector $v \in \mathbb{R}^n$, we define $v' = Tv^*/\| Tv^* \|$. Figure 13 illustrates color representations for various baryon graphs as well as some forbidden graphs which occur during decay processes. These representations show how the vertices move in color space during decay processes. They also show that in order to distinguish between strange and charmed vertices, the charmed vertices must have three (or more) petals.

Similar representations can be given for meson graphs. In order to distinguish between the vertex and antivertex of a meson, we assume that the color of an antivertex is the negative of the color for the corresponding vertex. This results in multiplying the coordinates of an antivertex by -1 in the color representation. Color representations for some meson graphs are shown in Figure 14.

The color space representation does not give a complete description of a particle, since it does not specify the spin and electric charge of a vertex, nor does it distinguish between vertices and antivertices. In order to give such a complete description we form the 48-dimensional hyperspace $\mathbb{R}^{48}$. Hyperspace is the tensor product

$$\mathbb{R}^{48} = \mathbb{R}^3 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^4$$

where $\mathbb{R}^3$ is the color space already considered, $\mathbb{R}^2$ is the spin space, the second $\mathbb{R}^2$ is the vertex-antivertex space, and $\mathbb{R}^4$ is the electric charge space. We take the usual standard bases for these spaces. In the spin space $\mathbb{R}^2$, $(1, 0)$ represents spin up and $(0, 1)$ spin down, In the vertex-antivertex space $\mathbb{R}^2$, $(1, 0)$ represent a vertex and $(0, 1)$ an antivertex. In the electric charge space $\mathbb{R}^4$, $(1, 0, 0, 0)$ represents electric charge $-1/3$ for vertices and

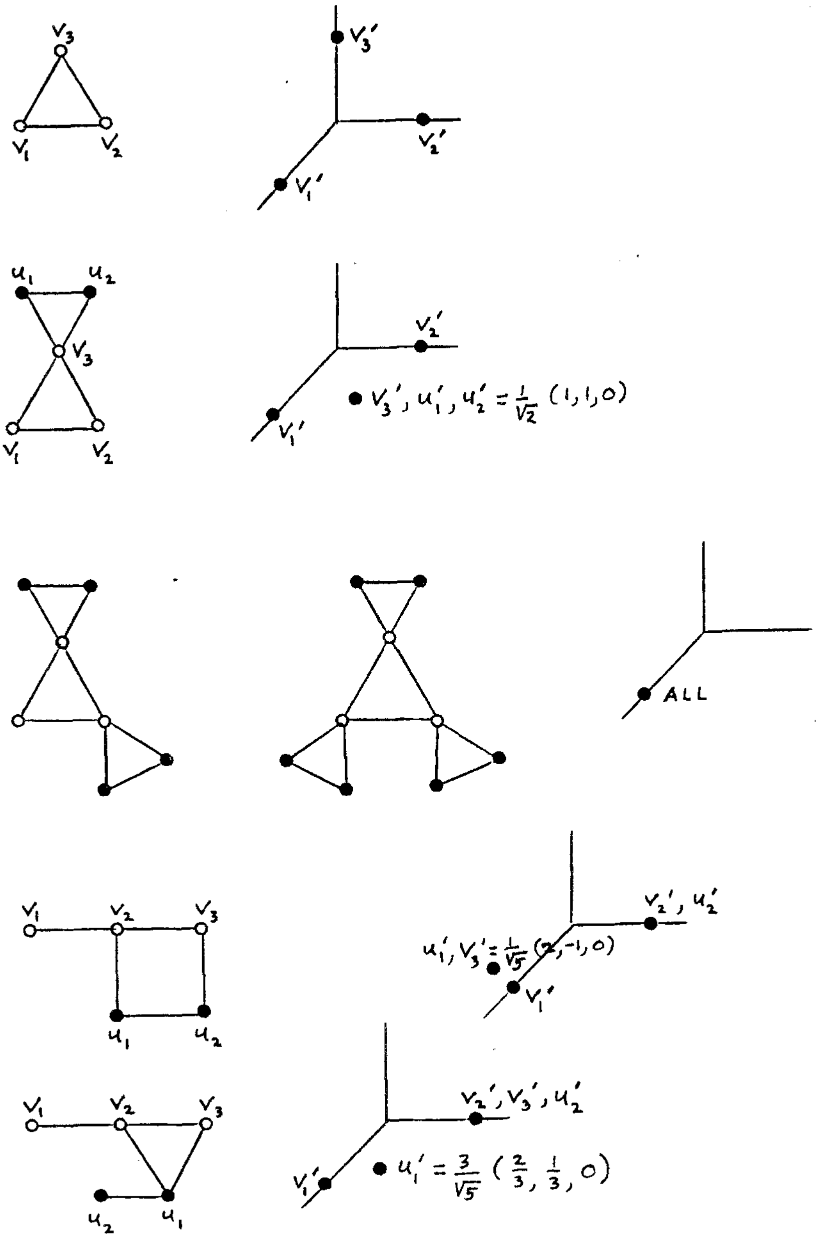

Fig. 13. Baryon color representations.

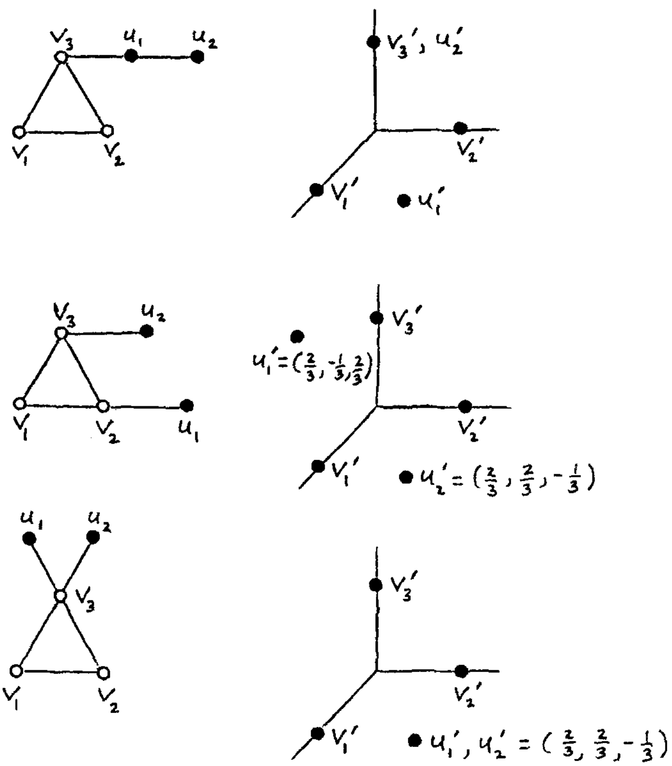

Fig. 13. Continued.

+1/3 for antivertices, (0, 1, 0, 0) represents +2/3 for vertices and −2/3 for antivertices, (0, 0, 1, 0) represents −1 for vertices and +1 for antivertices, and finally (0, 0, 0, 1) represents electric charge 0.

We can now construct the hyperspace representation of particles. For example, in hyperspace, the neutron is represented by three vectors,

$$\mathrm{n}=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\right.$$

$$\left.\begin{pmatrix}0\\0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right\}$$

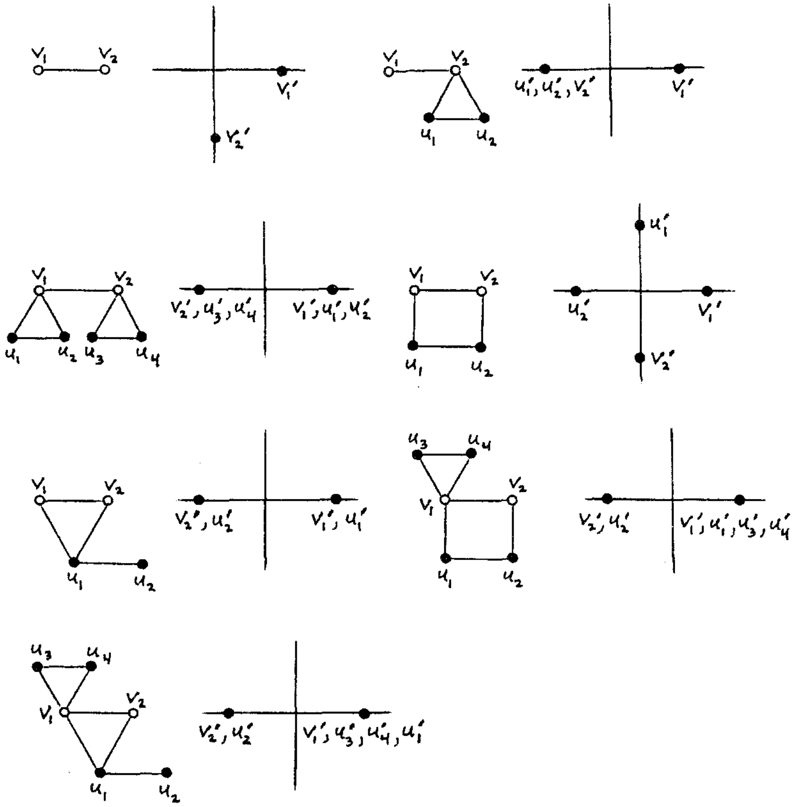

Fig. 14. Meson color representations.

The $\sum^-$ baryon is represented by five vectors,

$$\sum{}^{-} = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \right.$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$

$$\left. \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$$

The π^0 meson is represented by two vectors,

$$\pi^0 = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, -\begin{pmatrix}0\\1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \right\}$$

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